CFD Notes

Part I: CFD Foundations

Notation

$$\boldsymbol{V} = u\,\hat{\boldsymbol{\imath}} + v\,\hat{\boldsymbol{\jmath}} + w\,\hat{\boldsymbol{k}}, \qquad c = \frac{a\Delta t}{\Delta x}, \qquad d = \frac{\alpha \Delta t}{(\Delta x)^2}, \qquad G = \text{amplification factor.}$$

Introduction to Navier–Stokes Equations and Vector Algebra

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{V} + \frac{\boldsymbol{f}_m}{\rho} \tag{1}$$

$$\nabla \cdot \boldsymbol{V} = 0 \tag{2}$$

(1) $\boldsymbol{V} \cdot \nabla \boldsymbol{V}$ is a non-linear term involving the convection of velocity. Essentially, a fluid particle changes velocity as it moves from one place to another.

Imagine a drop of dye in a flowing stream. The quantity of dye is described by the local velocity field. A property field, such as velocity or pressure, can be transported rapidly depending on the flow. The field $\boldsymbol{V}$ is carried by the flow and has components

$$\boldsymbol{V} = u\,\hat{\boldsymbol{\imath}} + v\,\hat{\boldsymbol{\jmath}} + w\,\hat{\boldsymbol{k}}.$$

The term $\boldsymbol{V} \cdot \nabla\phi$ accounts for the fact that the property ϕ is transported along a fluid bulk velocity $\boldsymbol{V}$.

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \boldsymbol{V} \cdot \nabla \phi$$

So the quantity is advected by the fluid velocity.

(2) $\nabla \cdot \boldsymbol{V} = 0$ is the continuity equation for incompressible flow.

The momentum conservation formula,

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$

represents the movement of momentum through three dimensions via the velocity field.

The equation is first order in time. The convective term introduces first-order spatial derivatives.

First, note that we assume the fluid is Newtonian, so we assume μ is constant.

Fix ν. Also note that the Laplacian operator shows that the equation is second order in space:

$$\nu \nabla^2 \boldsymbol{V}.$$

$\nu \nabla^2 \boldsymbol{V}$ is the diffusion term. However, one must understand that this does not refer to mass diffusion from one region to another. This term describes the effect of internal friction in a fluid as adjacent fluid parcels have different velocities. The effect of friction is strongest near walls and in regions of strong velocity gradients. The quantity diffusing in the N–S equations is velocity, from high to low regions. Here, "diffusion" refers to the smoothing of velocity gradients due to viscosity.

Also recall that:

Divergence operator $\nabla\cdot$
Turns a vector into a scalar.

Gradient operator ∇
Turns a scalar into a vector:

$$\begin{Bmatrix} \dfrac{\partial \phi}{\partial x_1} \\ \vdots \\ \dfrac{\partial \phi}{\partial x_n} \end{Bmatrix}$$

Another interesting point:

If we say $\mu = 0$ in inviscid flow, then Re will be incredibly large (in essence, ∞), and N–S reduces to

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = -\frac{1}{\rho}\nabla p,$$

which are the Euler equations, and are still non-linear.

The fact that the N–S equations are both non-linear and second order in space makes them difficult to solve.

Consider an Infinitesimally Small Fluid Element Moving with Flow

Let

$$f_1 = f(x_1, y_1, z_1, t_1), \qquad f_2 = f(x_2, y_2, z_2, t_2).$$

Because the two states are on the same streamline, we may relate them with a Taylor series expansion. We view the fluid element as the smallest possible macroscopic representation of fluid, not as individual molecules. Only the first few terms of Taylor expansion are practical.

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)\Delta x^2}{2!} + \cdots + \frac{f^{(n)}(x)\Delta x^n}{n!}$$

Taking the first terms,

$$f_2 - f_1 = \left(\frac{\partial f}{\partial x}\right)(x_2 - x_1) + \left(\frac{\partial f}{\partial y}\right)(y_2 - y_1) + \left(\frac{\partial f}{\partial z}\right)(z_2 - z_1) + \left(\frac{\partial f}{\partial t}\right)(t_2 - t_1).$$

So

$$\frac{f_2 - f_1}{t_2 - t_1} = \left(\frac{\partial f}{\partial x}\right)\frac{x_2 - x_1}{t_2 - t_1} + \left(\frac{\partial f}{\partial y}\right)\frac{y_2 - y_1}{t_2 - t_1} + \left(\frac{\partial f}{\partial z}\right)\frac{z_2 - z_1}{t_2 - t_1} + \left(\frac{\partial f}{\partial t}\right).$$

Introduce the limit $t_2 \to t_1$:

$$\lim_{t_2 \to t_1} \frac{f_2 - f_1}{t_2 - t_1} \to \frac{Df}{Dt},$$

which is the time rate of change of a property in a Lagrangian frame.

Hence

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} = \frac{\partial f}{\partial t} + \boldsymbol{V} \cdot \nabla f.$$

The local derivative is instantaneous. The $\boldsymbol{V} \cdot \nabla f$ term is the rate of change due to spatial movement through the field.

This can be applied for finding u, v, w, T, ρ, and similar quantities.

Divergence of velocity: $\nabla \cdot \boldsymbol{V}$
Consider a moving control volume with fixed mass, invariant with time. This control volume and its shape will constantly change as it moves through regions with different velocities.

In Δt, only the normal component of $\boldsymbol{V}$ contributes to volume change:

$$(\boldsymbol{V} \cdot \hat{\boldsymbol{n}})\, dS\, \Delta t$$

which is distance moved times area.
Hence

$$\frac{\iint (\boldsymbol{V} \cdot d\boldsymbol{S})}{\Delta t} = \frac{\Delta \delta \bar{\mathcal{V}}}{\Delta t} \to \frac{D\delta \bar{\mathcal{V}}}{Dt}.$$

Let the shrinking control volume approach a fluid particle:

$$\frac{D\delta\bar{\mathcal{V}}}{Dt} = \iint_{\partial\delta\bar{\mathcal{V}}} \boldsymbol{V} \cdot d\boldsymbol{S},$$

and dividing by $\delta\bar{\mathcal{V}}$ gives

$$\nabla \cdot \boldsymbol{V} = \frac{1}{\delta\bar{\mathcal{V}}}\frac{D\delta\bar{\mathcal{V}}}{Dt}.$$

So $\nabla \cdot \boldsymbol{V}$ is the time rate of change of volume of a fluid element per unit fluid-element volume.

Mass Conservation

We derive the continuity equation first in Eulerian form and then in Lagrangian form.

For a fixed control volume in Eulerian space, net mass flow out of the control volume equals the time rate of decrease of mass inside the control volume.

Mass across a fixed surface S:

$$\dot{m} \text{ across element } = \rho \boldsymbol{V} \cdot d\boldsymbol{S}.$$

Therefore

$$\iint_S \rho \boldsymbol{V} \cdot d\boldsymbol{S} = -\frac{\partial}{\partial t} \iiint_{CV} \rho \, d\mathcal{V} \qquad (1)$$

The minus sign is there because if more mass leaves the fixed volume, the interior mass decreases.

By the divergence theorem,

$$\iiint_{CV} \nabla \cdot (\rho \boldsymbol{V}) \, d\mathcal{V} = -\frac{\partial}{\partial t} \iiint_{CV} \rho \, d\mathcal{V}.$$

Carry $\partial/\partial t$ inside:

$$\iiint_{CV} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) \right) d\mathcal{V} = 0.$$

Since the control volume is arbitrary, the integrand must be zero everywhere. Hence

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0}$$

which is the conservation PDE form.

For a moving fluid parcel, mass is fixed:

$$\frac{D(\delta m)}{Dt} = 0.$$

Since $\delta m = \rho \, \delta \bar{\mathcal{V}}$,

$$\frac{D(\rho \, \delta \bar{\mathcal{V}})}{Dt} = 0$$

so

$$\delta \bar{\mathcal{V}} \frac{D\rho}{Dt} + \rho \frac{D\delta \bar{\mathcal{V}}}{Dt} = 0.$$

Divide through by $\delta \bar{\mathcal{V}}$:

$$\frac{D\rho}{Dt} + \rho \left(\frac{1}{\delta \bar{\mathcal{V}}} \frac{D\delta \bar{\mathcal{V}}}{Dt} \right) = 0.$$

Recall

$$\nabla \cdot \boldsymbol{V} = \frac{1}{\delta \bar{\mathcal{V}}} \frac{D\delta \bar{\mathcal{V}}}{Dt},$$

hence

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{V} = 0}$$

which is the non-conservation PDE form, obtained by following a moving fluid parcel.

If

$$\frac{D\rho}{Dt} = 0,$$

i.e. incompressibility, then mass conservation reduces to

$$\boxed{\nabla \cdot \boldsymbol{V} = 0} \qquad \text{or} \qquad \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

Momentum Equation in the xx-Direction Ignoring Body Forces

Recall $\sum F_x = ma_x$. For a moving element:

$$a_x = \frac{Du}{Dt}, \qquad \sum F_x = m\frac{Du}{Dt}.$$

The mass of a fluid element is

$$m = \rho\,\delta x\,\delta y\,\delta z.$$

This figure shows the distribution of surface stresses consisting of shear and volumetric stresses.

Pressure contribution gives

$$-\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z.$$

Stress contributions give

$$\frac{\partial \tau_{xx}}{\partial x}\,\delta x\,\delta y\,\delta z + \frac{\partial \tau_{yx}}{\partial y}\,\delta x\,\delta y\,\delta z + \frac{\partial \tau_{zx}}{\partial z}\,\delta x\,\delta y\,\delta z.$$

Therefore

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}.$$

Stress can be written as a function of velocity. This comes from Newton's law of viscosity, i.e. linear deformation. Viscosity has two terms:

- shear deformation
- volumetric deformation

and two viscosities. In most cases, the effects of λ are small and so dilation effects are often ignored.

Use

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda(\nabla\cdot\boldsymbol{V}),$$

and

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \qquad \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right).$$

Substitute:

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x} + \lambda\nabla\cdot\boldsymbol{V}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right].$$

For constant μ,

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + (\mu+\lambda)\frac{\partial}{\partial x}(\nabla\cdot\boldsymbol{V}).$$

For incompressible flow,

$$\nabla\cdot\boldsymbol{V} = 0,$$

so

$$\rho\left(\frac{\partial u}{\partial t}+\boldsymbol{V}\cdot\nabla u\right)=-\frac{\partial p}{\partial x}+\mu\nabla^2 u.$$

Hence in vector form for incompressible flow,

$$\boxed{\rho\left(\frac{\partial \boldsymbol{V}}{\partial t}+\boldsymbol{V}\cdot\nabla \boldsymbol{V}\right)=-\nabla p+\mu\nabla^2 \boldsymbol{V}}$$

which is the incompressible momentum equation.

Basic Ingredients of CFD

For a solution, we need the full velocity field and the pressure field. The N–S equations have very few analytical solutions. Most analytical solutions exist only for very simple, highly symmetric flows where the non-linear term $\boldsymbol{V} \cdot \nabla \boldsymbol{V}$ reduces or vanishes.

Basic ingredients of CFD

(1) Mathematical model: set of PDEs? Integral? Differential?
Understand the physical model and which PDE should be used under the target application:

- incompressible
- inviscid
- turbulent
- 2D
- 3D

In aerospace, fluid motion arises from the collective motion of particles. We apply laws of motion and transport to fluids by first identifying the variables needed to define the resulting flow properties.

(2) Discretising model: a means of approximating the PDEs with a set of solvable algebraic equations:

$$\text{Continuous } f(x) \;\rightarrow\; \Delta t,\ \Delta x \text{ etc.}$$

There are many different methods of discretisation:

- Finite Difference
- Finite Volume
- Finite Element
- Particle methods

We must discretise because the computer works with numbers at discrete locations, not continuous fields.

Geometry / mesh / particles
The model assigns variables, operators, and equations at discrete nodes or control volumes.

(3) Analyse numerical scheme. Certain conditions must be met upon analysis, such as:

- stability
- convergence
- consistency

(4) Solve: obtain grid values of all flow variables.

Going back to (1): governing equations of fluid flow are mathematical statements of conservation laws of physics:

- Mass conserved: $\rho_1 V_1 = \rho_2 V_2$, or for incompressible flow $\nabla \cdot \boldsymbol{V} = 0$
- $\boldsymbol{F} = m\boldsymbol{a}$
- Energy is conserved

Before we apply these laws, we must pick a suitable model:

- Eulerian: fixed control volume, fluid moves through it, and changes in time create fluxes.
- Lagrangian: moving fluid particle; the particle moves while we follow it.

The integral forms are conservation forms, while the partial differential equations are differential forms.

Taylor Series

Taylor expansion is the basis for deriving finite-difference approximations.

If $f(x)$ is a continuous function of x, then $f(x + \Delta x)$ can be estimated provided we know $f(x)$:

$$f(x + \Delta x) \approx f(x)$$

(a) = initial guess

$$f(x + \Delta x) = f(x) + f'(x)\Delta x$$

(b) = captures slope

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)\Delta x^2}{2!}$$

(c) = captures curvature between the first two approximations

Adding more terms increases accuracy.

The Finite Difference method was introduced by Euler; it is one of the oldest and most straight-forward numerical methods for approximating partial derivatives using Taylor expansion.

After discretising a domain into a Cartesian grid, with nodes identified by a set of indices, e.g. (i, j) in 2D, recall that a derivative takes the form

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x_i + \Delta x) - u(x_i)}{\Delta x}.$$

Since we use x_i and $x_i + \Delta x$, this is a forward difference. But if there is $x_i - \Delta x$, it becomes a backward difference, and central difference can also be used.

BD

$$\frac{\partial u}{\partial x} \approx \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x)$$

FD

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1} - u_i}{\Delta x} + O(\Delta x)$$

CD

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2)$$

Central difference provides smaller truncation error.

More accurate finite-difference equations of third and fourth order can be used by taking information from more grid points. However, this increases the stencil width. For instance, a fourth-order FD scheme is

$$\frac{\partial u}{\partial x_i} = \frac{-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}}{12\Delta x} + O(\Delta x^4).$$

A drawback of higher-order schemes is that if the function is not smooth enough, Runge's phenomenon can occur, leading to oscillations and possible instability.

Derivation of Finite Difference Schemes

Write Taylor expansions about x_i:

$$u_{i+1} = u_i + \left.\frac{\partial u}{\partial x}\right|_{x_i} \Delta x + \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} \frac{\Delta x^2}{2!} + \left.\frac{\partial^3 u}{\partial x^3}\right|_{x_i} \frac{\Delta x^3}{3!} + \cdots \qquad (a)$$

$$u_{i-1} = u_i - \left.\frac{\partial u}{\partial x}\right|_{x_i} \Delta x + \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} \frac{\Delta x^2}{2!} - \left.\frac{\partial^3 u}{\partial x^3}\right|_{x_i} \frac{\Delta x^3}{3!} + \cdots \qquad (b)$$

Subtract u_i and divide by Δx:

$$\frac{u_{i+1} - u_i}{\Delta x} = \left.\frac{\partial u}{\partial x}\right|_{x_i} + O(\Delta x)$$

which is FD.

Similarly,

$$\frac{u_i - u_{i-1}}{\Delta x} = \left.\frac{\partial u}{\partial x}\right|_{x_i} + O(\Delta x)$$

which is BD.

Now write u_{i+1} and u_{i-1} on top of each other and subtract:

$$(a) - (b)$$

which gives

$$\frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2) = \left.\frac{\partial u}{\partial x}\right|_{x_i}.$$

So

$$\boxed{\left.\frac{\partial u}{\partial x}\right|_{x_i} \approx \frac{u_{i+1} - u_{i-1}}{2\Delta x}}$$

Second-order central first difference.

Now add:

$$(a) + (b)$$

which gives

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2) = \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i}.$$

Hence

$$\boxed{\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}}$$

Second-order central second difference.

1D Advection Equation

Writing the momentum equation in x-coordinates and reducing to 1D gives

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}.$$

If we ignore viscosity and the pressure field, this leaves

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

This wave is travelling in space at a velocity c. Here the field u is being carried by the flow at the average speed c.

One concern for the 1D advection equation is that the true domain is continuous, whereas computers only work with discrete points. At time step n we know u_i^n from the boundary and initial conditions, but for $c > 0$ we should not use downstream information such as u_{i+1}^n when the physical information travels from upwind to downwind. There is no reason to look forward in the domain if $c > 0$. Using downstream information can lead to non-physical behaviour and potential instability. We want an upwind interaction, from the direction of advection.

Use

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}, \qquad \frac{\partial u}{\partial x} \approx \frac{u_i^n - u_{i-1}^n}{\Delta x}.$$

Then

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0.$$

Transpose for u_i^{n+1}:

$$u_i^{n+1} = u_i^n - c\frac{\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right).$$

This is an explicit time-marching method; all the information on the RHS is known.

A Quick Side Note on the CFL Number

$$\text{CFL} = \frac{c\Delta t}{\Delta x}$$

Achieving a CFL below 1 is a key stability condition for many explicit schemes, amongst other factors such as mesh quality. This stems from the physical nature of hyperbolic PDEs. In explicit time-marching schemes, the CFL number quantifies the fraction of the cell size covered by the flow in one time step.

The Courant number is

$$\text{CFL} = \frac{c\Delta t}{\Delta x}.$$

If CFL < 1, the flow travels less than one grid spacing during a time step.
CFL is the ratio between this travel distance and the normal x-dimension.

More generally,

$$\text{CFL}_n = \Delta t \sum_{i=1}^{m} \frac{U_i}{\Delta x_i}.$$

When CFL > 1 (the exact CFL limit varies by scheme), the solution can blow up. This is because information propagates beyond one grid cell within a single time step, violating the numerical domain of dependence.

Reducing Δt improves this. Also note that if you reduce Δx, then you must also reduce Δt to maintain a similar CFL limit.

CFL limits time-step size in explicit solvers. An adaptive time-step approach is useful for complex flow fields where the speed is not constant.

Example:

$$nt = 40, \qquad \text{CFL} = 0.7, \qquad c = 2, \qquad \Delta x = 0.003$$

$$\Delta t = \frac{(\Delta x)(\text{CFL})}{c} = \frac{0.003 \times 0.7}{2} = 0.00105.$$

Modified Equation and Numerical Viscosity

Start from

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+c\frac{u_i^n-u_{i-1}^n}{\Delta x}=0 \qquad (1)$$

Expand term (a):

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}\frac{\Delta t}{2}+\cdots+O(\Delta t^2)$$

Expand term (b):

$$\frac{u_i^n-u_{i-1}^n}{\Delta x}=\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}\frac{\Delta x}{2}+\cdots+O(\Delta x^2)$$

Substitute into (1):

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}\frac{\Delta t}{2}+c\frac{\partial u}{\partial x}-c\frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}+\cdots=0 \qquad (2)$$

From (1),

$$\frac{\partial u}{\partial t}=-c\frac{\partial u}{\partial x}.$$

Differentiate with respect to t:

$$\frac{\partial^2 u}{\partial t^2}=-c\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right).$$

Assuming smoothness, swap derivatives:

$$\frac{\partial^2 u}{\partial t^2}=-c\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right).$$

Substitute

$$\frac{\partial u}{\partial t}=-c\frac{\partial u}{\partial x}$$

to obtain

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}.$$

Rewrite (2):

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}+\frac{\Delta t}{2}c^2\frac{\partial^2 u}{\partial x^2}-c\frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}+\cdots=0$$

so

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=\left(\frac{c\Delta x}{2}-\frac{c^2\Delta t}{2}\right)\frac{\partial^2 u}{\partial x^2}+\cdots$$

Factor:

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=\frac{c}{2}(\Delta x-c\Delta t)\frac{\partial^2 u}{\partial x^2}+\cdots$$

Since

$$\text{CFL}=\frac{c\Delta t}{\Delta x},$$

this becomes

$$\boxed{\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=\frac{c\Delta x}{2}(1-\text{CFL})\frac{\partial^2 u}{\partial x^2}+\cdots} \qquad (3)$$

(3) is a modified equation.

If CFL > 1, the scheme becomes unstable. If CFL < 1, the viscosity term smooths sharp fronts. If CFL $= 1$, it removes the viscous term.

The coefficient

$$\mu_{\text{num}} = \frac{c\Delta x}{2}(1 - \text{CFL})$$

is the numerical viscosity.

Replacing $\partial^2 u/\partial t^2$ with $c^2 \partial^2 u/\partial x^2$ from the original equation is called the Cauchy–Kowalewsky procedure. Deriving a modified equation of a given scheme can show what the scheme is actually solving and provide deep insight into the behaviour of the numerical solution.

For example, FTCS introduces negative numerical viscosity and is unconditionally unstable.

In principle, we can only hope that the numerical result gives values for u which represent those that would be obtained from a closed-form analytical solution.

Part II: Errors and Stability Analysis

Errors and Stability Analysis

We now analyse how numerical errors propagate and affect stability.

The main reason why explicit methods become numerically unstable is that the increment in the marching direction exceeds a prescribed value. Determining this value comes from a formal analysis of the finite-difference scheme. It is not always necessary to go through a completely rigorous stability test, but an overview of two methods applied to relatively simple PDEs is useful.

Calculations are said to be stable if error decreases from one time step to another. So our main task in stability analysis is to determine the manner in which an error propagates throughout a solution.

Two types of errors:

1) **Discretisation Error**, which is simply the truncation error introduced from the Taylor expansion. It must be noted that the leading-order truncation error term often appears in front of a second derivative term and acts like *artificial viscosity*; it smooths gradients.

2) **Round-Off Error**, introduced after a large number of calculations in which the computer is continually rounding values.

$$A \to \text{analytical solution of PDE}$$

$$D \to \text{exact solution of FDE}$$

$$N \to \text{numerical solution from a real computer with finite accuracy}$$

$$N = D + \varepsilon, \qquad \phi_{\text{disc}} = A - D$$

Since N is the output from the FDE, it must satisfy the difference equation. Take, for instance, the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad \text{using FTCD} \tag{1}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{2}$$

Round-Off Error Satisfies the Difference Equation

$$U_i^n = D_i^n + \varepsilon_i^n$$

Substituting into the finite-difference equation:

$$\frac{(D_i^{n+1} + \varepsilon_i^{n+1}) - (D_i^n + \varepsilon_i^n)}{\Delta t} = \alpha \frac{(D_{i+1}^n + \varepsilon_{i+1}^n) - 2(D_i^n + \varepsilon_i^n) + (D_{i-1}^n + \varepsilon_{i-1}^n)}{(\Delta x)^2} \tag{3}$$

D is the exact solution of the FDE and satisfies it:

$$\frac{D_i^{n+1} - D_i^n}{\Delta t} = \alpha \frac{D_{i+1}^n - 2D_i^n + D_{i-1}^n}{(\Delta x)^2}$$

Rearrange (3) to get:

$$\frac{D_i^{n+1} - D_i^n}{\Delta t} = \alpha \frac{D_{i+1}^n - 2D_i^n + D_{i-1}^n}{(\Delta x)^2} + \alpha \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{(\Delta x)^2} - \frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t}$$

Hence,

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \alpha \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{(\Delta x)^2}$$

As you can see, the error (round-off) satisfies the difference equation. This makes sense: if U satisfies it and so does D, then ε must also, since U is the sum of D and ε.

Let us also introduce another term:

$$\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n}$$

This is the **amplification factor** G; it tells us how much the error has grown in one time step.
Suppose $G = 1$, so

$$\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = 1,$$

then the error has not grown at all, and we need not worry about how the round-off error propagates. But if

$$\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} > 1,$$

it will continue to grow to ∞. To prevent this, we must bound the amplification factor:

$$|G| \leq 1.$$

The amplification factor shows that the growth or decay of an error can be exponential. This will be important in the next section on Von-Neumann stability analysis.

Fourier Representation of Error

The random variation of error can be expressed as a Fourier series, i.e. a sequential sum of waves with increasing wave modes, each representing a higher harmonic.

$$\varepsilon(x) = \sum_m A_m e^{ik_m x}$$

Here k_m is the wave number, and

$$e^{ik_m x} = \cos(k_m x) + i \sin(k_m x).$$

n_x is the number of grid points.

Total grid intervals:

$$N = n_x - 1, \qquad \Delta x = \frac{L}{N} = \frac{L}{n_x - 1}.$$

Let us consider a sine function. We know that λ is the distance over which a wave repeats, and $\sin(x)$ repeats every 2π. To get this in wave-number notation, we want to shrink the $\sin(x)$ by 2π and then stretch it by λ:

$$\sin\left(\frac{2\pi x}{\lambda}\right) \to \sin(k_m x), \qquad k_m = \frac{2\pi}{\lambda}.$$

Let us show the significance of m.

If one entire sine wave fits in the domain, it is expressed by $\sin(k_m x)$ where

$$k_m = \frac{2\pi}{L}.$$

If 2 sine waves fit, then $\lambda = L/2$ and

$$k_m = \left(\frac{2\pi}{L}\right) 2.$$

If 3 fit, then

$$k_m = \left(\frac{2\pi}{L}\right) 3.$$

So the wave number of sine waves is

$$k_m = \left(\frac{2\pi}{L}\right) m, \qquad m = 1, 2, 3, \ldots$$

In

$$\sum_m A_m e^{ik_m x},$$

as if $n \to \infty$ then the Fourier series can represent any continuous function.

For a numerical domain of length L, the largest wavelength occurs at $m = 1$:

$$\sin(k_1 x), \qquad k_1 = \frac{2\pi}{L}.$$

The smallest wavelength occurs when all three zeros of a sine wave pass through three adjacent grid points:

$$i-1, \quad i, \quad i+1.$$

Hence

$$\lambda = 2\Delta x = 2\left(\frac{L}{N}\right),$$

so

$$k_{\max} = \frac{2\pi}{\lambda} = \frac{2\pi}{2L/N} = \left(\frac{2\pi}{L}\right)\frac{N}{2}.$$

So in a numerical scheme, the Fourier series has an upper bound of $N/2$:

$$\varepsilon(x) = \sum^{N/2} A_m e^{ik_m x}.$$

Von-Neumann Analysis Procedure

It is also logical to assume that $\varepsilon(x)$ is also a function of time:

$$\varepsilon(x,t) = \sum_{m}^{N/2} A_m(t) e^{ik_m x}$$

Take

$$A_m(t) \to e^{at},$$

which is a reasonable assumption.
Then

$$\varepsilon(x,t) = \sum_{m}^{N/2} e^{at} e^{ik_m x}.$$

We now have a truncated Fourier series with exponential amplitudes (which may have different values for different modes) that vary with time.

Recalling the heat equation:

$$L(\cdot) \equiv \frac{\partial}{\partial t} - \alpha \frac{\partial^2}{\partial x^2} = 0$$

that is,

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad \Longrightarrow \quad \frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = 0.$$

This is a linear operator and because of that you can apply the operator to individual modes in the Fourier series, using the principle of superposition, since a Fourier series is a summation of the time evolution of modes.

$$L(\varepsilon(x,t)) = L\left(\sum_{m}^{N/2} e^{at} e^{ik_m x}\right) = \sum_{m}^{N/2} L\left(e^{at} e^{ik_m x}\right)$$

This shows that we can apply the linear operator to each term to study individual error-mode behaviour without losing generality.

This introduces the procedure of **Von-Neumann analysis**:

1) Deal with the behaviour of a single error term:

$$\varepsilon_i^n = e^{at} e^{ik_m x}.$$

2) Substitute $e^{at} e^{ik_m x}$ for ε_i^n in a FDE.

3) Divide by $e^{at} e^{ik_m x}$.

4) Obtain G, which is $e^{a\Delta t}$ since

$$\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = \frac{e^{a(t+\Delta t)} e^{ik_m x}}{e^{at} e^{ik_m x}} = e^{a\Delta t}.$$

5) Bound

$$|e^{a\Delta t}| \leq 1,$$

and determine the nature of the resulting expression.

Example: Back to the Heat Equation FTCD

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \alpha \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{(\Delta x)^2}$$

Replace with $\varepsilon_i^n = e^{at} e^{ik_m x}$:

$$\frac{e^{a(t+\Delta t)} e^{ik_m x} - e^{at} e^{ik_m x}}{\Delta t} = \alpha \frac{e^{at} e^{ik_m (x+\Delta x)} - 2e^{at} e^{ik_m x} + e^{at} e^{ik_m (x-\Delta x)}}{(\Delta x)^2}$$

Dividing by $e^{at} e^{ik_m x}$ and rearranging for $e^{a\Delta t}$ gives

$$e^{a\Delta t} = 1 + \frac{\alpha \Delta t}{(\Delta x)^2} \left(e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x} \right)$$

Knowing

$$e^{i\delta} = \cos\delta + i\sin\delta, \qquad e^{-i\delta} = \cos\delta - i\sin\delta,$$

and

$$d = \frac{\alpha \Delta t}{(\Delta x)^2},$$

we get

$$G = e^{a\Delta t} = 1 + 2d\left(\cos(k_m \Delta x) - 1\right).$$

For stability,

$$|G| \leq 1.$$

Since $\cos(k_m \Delta x) \in [-1, 1]$:

- If $\cos(k_m \Delta x) = 1$, then $G = 1$, always satisfied.
- If $\cos(k_m \Delta x) = -1$, then

  $$G = 1 - 4d.$$

  So we need

  $$-1 \leq 1 - 4d \leq 1.$$

The right inequality is always satisfied. The left gives:

$$-1 \leq 1 - 4d \quad \Longrightarrow \quad -2 \leq -4d \quad \Longrightarrow \quad d \leq \frac{1}{2}.$$

Hence the stability condition is

$$\boxed{\frac{\alpha \Delta t}{(\Delta x)^2} \leq 0.5}$$

FTCD on the 1D Advection PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$

Hence,

$$u_i^{n+1} = u_i^n - \frac{c}{2}(u_{i+1}^n - u_{i-1}^n), \qquad c = \frac{a\Delta t}{\Delta x}.$$

Therefore,

$$\varepsilon_i^{n+1} = \varepsilon_i^n - \frac{c}{2}(\varepsilon_{i+1}^n - \varepsilon_{i-1}^n)$$

Replace $\varepsilon_i^n = e^{at}e^{ik_m x}$:

$$e^{a(t+\Delta t)}e^{ik_m x} = e^{at}e^{ik_m x} - \frac{c}{2}\left(e^{at}e^{ik_m(x+\Delta x)} - e^{at}e^{ik_m(x-\Delta x)}\right)$$

Thus

$$e^{a\Delta t} = 1 - \frac{c}{2}\left(e^{ik_m\Delta x} - e^{-ik_m\Delta x}\right)$$

Hence

$$e^{a\Delta t} = 1 - ci\sin(k_m\Delta x)$$

So

$$G = 1 - ci\sin(k_m\Delta x)$$

To assess stability, we consider the magnitude of the amplification factor:

$$|G|^2 = 1 + c^2\sin^2(k_m\Delta x)$$

For stability,

$$|G|^2 \le 1 \quad \Longrightarrow \quad 1 + c^2\sin^2(k_m\Delta x) \le 1.$$

If $\sin^2(k_m\Delta x) = 0$, then

$$1 + 0 = 1,$$

which is fine.
If $\sin^2(k_m\Delta x) = 1$, then

$$1 + c^2 \le 1,$$

which is impossible unless $c = 0$.
Since c is generally positive, then $1 + c^2 > 1$ always, and the scheme is

$$\boxed{\text{unconditionally unstable.}}$$

Sometimes a plot of G with varying values of c or d can also reveal the limits of stability.

Discrete Perturbation Stability Analysis

There is another, more cumbersome procedure for determining the stability limit of linear PDEs. It is called **Discrete Perturbation Stability Analysis** and involves introducing an error ε at a point in the solution and investigating its effect on the next time step at that point or even at neighbouring points.

So we first say the solution u at all points at time n is 0, but then introduce ε at u_i^n. Then we determine the solution at the next time step. Consider the 1D heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad \text{FTCD}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

Set the solution at time level n to zero everywhere except at node i, where the perturbation is ε:

$$u_i^n = \varepsilon, \qquad u_{i\pm1}^n = 0.$$

Then

$$\frac{u_i^{n+1} - \varepsilon}{\Delta t} = \alpha \frac{0 - 2\varepsilon + 0}{(\Delta x)^2}$$

so

$$u_i^{n+1} = \varepsilon + d(-2\varepsilon) = \varepsilon - 2d\varepsilon,$$

where

$$d = \frac{\alpha \Delta t}{(\Delta x)^2}.$$

Since the solution is 0, u_i^{n+1} is simply the propagation of the introduced error term:

$$\frac{u_i^{n+1}}{\varepsilon} = 1 - 2d$$

so

$$G_i = 1 - 2d.$$

Then

$$1 - 2d \leq 1$$

is always satisfied since $d > 0$,
and

$$1 - 2d \geq -1 \quad \Longrightarrow \quad d \leq 1.$$

We now aim to determine the effect of the initial disturbance at nodes $(i+1)$, $(i+2)$, $(i-1)$, $(i-2)$ at time step $(n+1)$.
It is found that for u_i^{n+2},

$$\frac{u_i^{n+2} - u_i^{n+1}}{\Delta t} = \alpha \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}$$

and so

$$u_i^{n+2} = \varepsilon_i^n (6d^2 - 4d + 1)$$

and

$$d \leq \frac{2}{3}.$$

This analysis is more involved and requires further development.

Further Practical Stability Considerations

To determine any practical stability criterion, we must consider the behaviour of error at later stages.

We may see that propagation of the error is of a smoothing type: it spreads and slowly reduces.

Two things to consider

1) Ideally, at time step m, the initial error will have reached every grid point in the solution and smoothed out to the same approximate value ε_m.

   Let us determine ε at $(m+1)$ now.

$$\frac{u_i^{m+1} - u_i^m}{\Delta t} = \alpha \frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{(\Delta x)^2}$$

   hence

$$u_i^{m+1} = u_i^m + d(u_{i+1}^m - 2u_i^m + u_{i-1}^m)$$

   Taking

$$u_{i+1}^m = u_i^m = u_{i-1}^m = \varepsilon_m,$$

   then

$$u_i^{m+1} = \varepsilon_m + d(\varepsilon_m - 2\varepsilon_m + \varepsilon_m) = \varepsilon_m.$$

   No stability criterion is imposed.

2) At time step m, error may display oscillatory motion with alternating signs:

$$u_{i+1}^m = -\varepsilon_m, \qquad u_i^m = \varepsilon_m, \qquad u_{i-1}^m = -\varepsilon_m.$$

   Then

$$u_i^{m+1} = \varepsilon_m + d(-\varepsilon_m - 2\varepsilon_m - \varepsilon_m) = \varepsilon_m - 4d\varepsilon_m$$

   so

$$\frac{u_i^{m+1}}{\varepsilon_m} = 1 - 4d.$$

   Therefore,

$$G = 1 - 4d, \qquad -1 \le 1 - 4d \le 1.$$

   The left inequality gives

$$d \le 0.5.$$

$$\boxed{d \le 0.5}$$

   We found this earlier with Von-Neumann analysis in a faster manner.

Analysing Stability Series

Here I will practise using Von-Neumann analysis on some 1D PDEs.

(1) 1D Advection FTBD

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_i^n - u_{i-1}^n}{\Delta x}$$

so

$$u_i^{n+1} = u_i^n - c(u_i^n - u_{i-1}^n)$$

Via Von-Neumann stability analysis:

$$e^{a\Delta t} = 1 - c\,(1 - \cos(k_m\Delta x) + i\sin(k_m\Delta x))$$

Hence

$$G = 1 + c\,(\cos(k_m\Delta x) - 1) - ic\sin(k_m\Delta x)$$

Let

$$z = \cos(k_m\Delta x).$$

Then

$$|G|^2 = [1 + c(z-1)]^2 + c^2\sin^2(k_m\Delta x)$$

Expand:

$$|G|^2 = 1 + 2c(z-1) + c^2(z^2 - 2z + 1) + c^2\sin^2(k_m\Delta x)$$

Using $\sin^2(k_m\Delta x) = 1 - z^2$,

$$|G|^2 = 1 + 2c(z-1) + c^2(z^2 - 2z + 1) + c^2(1 - z^2)$$

so

$$|G|^2 = 1 + 2c(z-1) + 2c^2(1-z)$$

For stability,

$$|G|^2 \le 1 \quad \Longrightarrow \quad 2c(z-1) + 2c^2(1-z) \le 0.$$

Check extremes:

- If $z = -1$:
$$-4c + 4c^2 \le 0 \quad \Longrightarrow \quad 4c^2 \le 4c \quad \Longrightarrow \quad c \le 1.$$
- If $z = 1$:
$$0 \le 0$$
  always satisfied.

Therefore,

$$\boxed{c \le 1.}$$

One may also convert the 1D advection PDE into a modified PDE to examine the effects of $c > 1$.

FTBD via Taylor Expansion / Modified PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad \text{FTBD}$$

It may also be shown via Taylor expansion.

Expand:

$$u(x+\Delta x) = u(x) + \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{\Delta x^2}{2} + O(\Delta x^3)$$

$$u(t+\Delta t) = u(t) + \frac{\partial u}{\partial t}\Delta t + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t^2}{2} + O(\Delta t^3)$$

Thus

$$\frac{u(t+\Delta t) - u(t)}{\Delta t} = \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t}{2} + O(\Delta t^2) \tag{1}$$

Also,

$$\frac{u(x) - u(x-\Delta x)}{\Delta x} = \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2}\frac{\Delta x}{2} + O(\Delta x^2) \tag{2}$$

So the discrete equation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

becomes

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta x^2, \Delta t^2) = 0$$

Let us get rid of $\partial^2 u/\partial t^2$ by recalling

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}.$$

Differentiate again:

$$\frac{\partial^2 u}{\partial t^2} = -a\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = -a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = -a\frac{\partial}{\partial x}\left(-a\frac{\partial u}{\partial x}\right) = a^2\frac{\partial^2 u}{\partial x^2}.$$

Hence

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \frac{a\Delta x}{2}(1-c)\frac{\partial^2 u}{\partial x^2} + O(a\Delta x^2, \Delta t^2)$$

where

$$c = \frac{a\Delta t}{\Delta x}.$$

The coefficient

$$\frac{a\Delta x}{2}(1-c) = \frac{a}{2}(\Delta x - a\Delta t)$$

acts like **artificial viscosity**.

If $c > 1$, then the viscosity coefficient becomes negative and the scheme is unstable. This is another way to demonstrate the stability criterion for hyperbolic PDEs.

1D Advection FTBD with Lax Method

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad \text{FTBD with Lax Method}$$

$$\frac{u_i^{n+1} - \frac{1}{2}(u_{i+1}^n + u_{i-1}^n)}{\Delta t} = -a\frac{u_i^n - u_{i-1}^n}{\Delta x}$$

So

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{a\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$$

Substitute a Fourier mode:

$$e^{a\Delta t} = \frac{1}{2}\left(e^{ik_m\Delta x} + e^{-ik_m\Delta x}\right) - c\left(1 - e^{-ik_m\Delta x}\right)$$

$$e^{a\Delta t} = \frac{1}{2}(2\cos(k_m\Delta x)) - c\left(1 - \cos(k_m\Delta x) + i\sin(k_m\Delta x)\right)$$

$$e^{a\Delta t} = \cos(k_m\Delta x) - c + c\cos(k_m\Delta x) - ic\sin(k_m\Delta x)$$

Thus

$$G = c[\cos(k_m\Delta x) - 1] + \cos(k_m\Delta x) - ic\sin(k_m\Delta x)$$

Then

$$|G|^2 = \left(c[\cos(k_m\Delta x) - 1] + \cos(k_m\Delta x)\right)^2 + c^2\sin^2(k_m\Delta x)$$

For simplicity denote

$$\delta = k_m\Delta x.$$

Then

$$|G|^2 = \left(c(\cos\delta - 1) + \cos\delta\right)^2 + c^2\sin^2\delta$$

Expanding:

$$|G|^2 = c^2(\cos\delta - 1)^2 + 2c(\cos\delta - 1)\cos\delta + \cos^2\delta + c^2\sin^2\delta$$

$$= c^2(\cos^2\delta - 2\cos\delta + 1) + 2c(\cos^2\delta - \cos\delta) + \cos^2\delta + c^2\sin^2\delta$$

Using $\cos^2\delta + \sin^2\delta = 1$:

$$|G|^2 = 2c^2 - 2c^2\cos\delta + 2c\cos^2\delta - 2c\cos\delta + \cos^2\delta$$

Let

$$\cos\delta = z,$$

then

$$|G|^2 = (1 + 2c)z^2 + (-2c^2 - 2c)z + 2c^2$$

For stability:

$$|G|^2 \leq 1, \qquad z \in [-1, 1].$$

Check endpoints:

1) $z = 1$:

$$1 + 2c - 2c^2 - 2c + 2c^2 \leq 1 \quad \Longrightarrow \quad 1 \leq 1$$

always satisfied.

2) $z = -1$:

$$1 + 2c + 2c^2 + 2c + 2c^2 \leq 1$$

$$4c^2 + 4c \leq 0 \quad \Longrightarrow \quad 2c(c+1) \leq 0$$

so

$$-1 \leq c \leq 0.$$

For $a > 0$, the stability condition cannot be satisfied in the useful range, so the scheme is unstable.